SUGGESTED SOLUTIONS TO HOMEWORK I

Solution 1. (1) Denote

$$\Gamma(x,y,z;\xi,\eta,\zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}},$$

then the Green's function to Ω is

$$\begin{aligned}G(x,y,z;\xi,\eta,\zeta) =&\Gamma(x,y,z;\xi,\eta,\zeta) - \Gamma(-x,y,z;\xi,\eta,\zeta) - \Gamma(x,-y,z;\xi,\eta,\zeta) - \Gamma(x,y,-z;\xi,\eta,\zeta)\\ &+ \Gamma(-x,-y,z;\xi,\eta,\zeta) + \Gamma(x,-y,-z;\xi,\eta,\zeta) + \Gamma(-x,y,-z;\xi,\eta,\zeta)\\ &- \Gamma(-x,-y,-z;\xi,\eta,\zeta).\end{aligned}$$

(2) Denote

$$\Gamma(x,y,z;\xi,\eta,\zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}},$$

then the Green's function to Ω is

$$\begin{aligned}G(x,y,z;\xi,\eta,\zeta) =&\Gamma(x,y,z;\xi,\eta,\zeta) - \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(x^*,y^*,z^*;\xi,\eta,\zeta)\\ &- \Gamma(-x,y,z;\xi,\eta,\zeta) + \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(-x^*,y^*,z^*;\xi,\eta,\zeta)\\ &- \Gamma(x,-y,z;\xi,\eta,\zeta) + \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(x^*,-y^*,z^*;\xi,\eta,\zeta)\\ &- \Gamma(x,y,-z;\xi,\eta,\zeta) + \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(x^*,y^*,-z^*;\xi,\eta,\zeta)\\ &+ \Gamma(-x,-y,z;\xi,\eta,\zeta) - \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(-x^*,-y^*,z^*;\xi,\eta,\zeta)\\ &+ \Gamma(x,-y,-z;\xi,\eta,\zeta) - \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(x^*,-y^*,-z^*;\xi,\eta,\zeta)\\ &+ \Gamma(-x,y,-z;\xi,\eta,\zeta) - \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(-x^*,y^*,-z^*;\xi,\eta,\zeta)\\ &- \Gamma(-x,-y,-z;\xi,\eta,\zeta) + \frac{R}{\sqrt{x^2+y^2+z^2}}\Gamma(-x^*,-y^*,-z^*;\xi,\eta,\zeta).\end{aligned}$$

Solution 2. (1) The solution is

$$u(t,x) = t^3 + e^{-t}\sin x.$$

(2) The solution is

$$u(t,x) = 1 + \mathcal{E}(\frac{x}{\sqrt{4kt}}),$$

where $\mathcal{E}$ is the Gauss error function,

$$\mathcal{E}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds.$$

Solution 3. (1) For arbitrary $x \in \Omega$, denote

$$\phi(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y) dS_y,$$

then

$$\begin{aligned} \frac{d\phi(r)}{dr} =& \frac{d}{dr} \left(\frac{1}{|\partial B_1|} \int_{\partial B_1(0)} u(x+rz) dS_z \right) \\ =& \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} \nabla u(x+rz) \cdot z dS_z \\ =& \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} \nabla u(y) \cdot \frac{y-x}{r} dS_y, \end{aligned}$$

since $\frac{y-x}{r}$ is an outer normal vector at $\partial B_r(x)$, then by Gauss's formula,

$$\frac{d\phi(r)}{dr} = \frac{r}{n|B_r|} \int_{B_r(x)} \Delta u(y) dy \geq 0,$$

which indicates that $\phi(r)$ is an increasing function. Then by the Lebesgue's theorem,

$$u(x) = \lim_{r \to 0} \phi(r) \leq \phi(r),$$

therefore

$$\begin{aligned} \frac{1}{|B_r|} \int_{B_r(x)} u(y) dy =& \frac{1}{|B_r|} \int_0^r \int_{\partial B_\rho(x)} u(y) dS_y d\rho \\ =& \frac{1}{|B_r|} \int_0^r |\partial B_\rho| \phi(\rho) d\rho \\ \geq& u(x). \end{aligned}$$

(2) Let $x_0 \in \overline{\Omega}$ be a point such that $u(x_0) = \sup u$, and assume that u is not identically equal to $u(x_0)$. If $x_0 \in \Omega$, then on the one hand, since for every ball $B_r(x_0) \subset \Omega$,

$$u(y) \leq u(x_0),$$

for all $y \in B_r(x_0)$, we have

$$\frac{1}{|B_r|} \int_{B_r(x_0)} u(y) dy \leq u(x_0).$$

On the other hand, by (1),

$$u(x_0) \leq \frac{1}{|B_r|} \int_{B_r(x_0)} u(y) dy,$$

which implies

$$u(x_0) = \frac{1}{|B_r|} \int_{B_r(x_0)} u(y) dy,$$

and $u(y) = u(x_0)$ for all $y \in B_r(x_0)$. Consider $\Omega_0 = \{y \in \Omega : u(y) = u(x_0)\}$, then by the above discussion, Ω_0 is open in Ω. In addition, because u is continuous, then Ω_0 is also closed in Ω. Since Ω is connected, therefore $\Omega_0 = \Omega$, which is a contradiction! Therefore $x_0 \in \partial\Omega$.

(3) Since u is a subharmonic function, by (2),

$$\max_{\overline{\Omega}} u = \max_{x^2+y^2=1}(3-5xy) = \frac{11}{2}.$$

Solution 4. (1) For arbitrary $t>0$ and $x\in\Omega$, denote

$$\phi(r) = \frac{1}{4r^n}\iint_{E_r(t,x)} u(s,y)\frac{|x-y|^2}{(t-s)^2}dyds,$$

then

$$\begin{aligned}\frac{d\phi(r)}{dr} =& \frac{d}{dr}\left(\frac{1}{4}\iint_{E_1(0,0)} u(t+r^2\tau, x+rz)\frac{|z|^2}{\tau^2}dzd\tau\right)\\ =& \frac{1}{4}\iint_{E_1(0,0)}(2r\tau\partial_s u(t+r^2\tau,x+rz)+\nabla_y u(t+r^2\tau,x+rz)\cdot z)\frac{|z|^2}{\tau^2}dzd\tau\\ =& \frac{1}{4r^{n+1}}\iint_{E_r(t,x)}(2(t-s)\partial_s u(s,y)+\nabla_y u(s,y)\cdot(x-y))\frac{|x-y|^2}{(t-s)^2}dyds\\ :=& A+B.\end{aligned}$$

Introduce

$$\psi(s,y) = -\frac{n}{2}\log(4\pi(t-s)) - \frac{|x-y|^2}{4(t-s)} + n\log r,$$

since

$$\partial_s\psi = \frac{n}{2(t-s)} - \frac{|x-y|^2}{4(t-s)^2},\quad \partial_{y_i}\psi = -\frac{(x_i-y_i)}{2(t-s)},$$

and $\psi(s,y)=0$ on $\partial E(t,x,r)$, then by Gauss's formula,

$$\begin{aligned}A =& \frac{1}{r^{n+1}}\iint_{E_r(t,x)}\partial_s u(s,y)\cdot(y\cdot\nabla_y\psi(s,y))dyds\\ =& -\frac{1}{r^{n+1}}\iint_{E_r(t,x)} n\partial_s u(s,y)\cdot\psi(s,y)+\nabla_y\partial_s u(s,y)\cdot y\cdot\psi(s,y)dyds\\ =& -\frac{1}{r^{n+1}}\iint_{E_r(t,x)} n\partial_s u(s,y)\cdot\psi(s,y)-\nabla_y u(s,y)\cdot y\cdot\partial_s\psi(s,y)dyds\\ =& -\frac{1}{r^{n+1}}\iint_{E_r(t,x)} n\partial_s u(s,y)\cdot\psi(s,y)-\frac{n}{2(t-s)}\nabla_y u(s,y)\cdot ydyds - B,\end{aligned}$$

therefore from the fact that u is a subsolution of the heat equation, by Gauss's formula,

$$\begin{aligned}\frac{d\phi(r)}{dr} \ge& -\frac{1}{r^{n+1}}\iint_{E_r(t,x)} n\Delta_y u(s,y)\cdot\psi(s,y)-\frac{n}{2(t-s)}\nabla_y u(s,y)\cdot ydyds\\ \ge& 0.\end{aligned}$$

Hence by the Lebesgue's theorem,

$$u(t,x) = \lim_{r\to 0}\phi(r) \le \phi(r).$$

(2) Let $(t_0,x_0)\in[0,T]\times\overline{\Omega}$ be a point such that $u(t_0,x_0)=\sup u$, and assume that u is not identically equal to $u(t_0,x_0)$. If $(t_0,x_0)\in(0,T]\times\Omega$, then on the one hand, since for every ball $E_r(t_0,x_0)\subset(0,T]\times\Omega$,

$$u(s,y)\le u(t_0,x_0),$$

for all $(s, y) \in E_r(t_0, x_0)$, we have

$$\frac{1}{4r^n} \iint_{E_r(t,x)} u(s,y) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds \leq u(t_0 x_0).$$

On the other hand, by (1),

$$u(t_0, x_0) \leq \frac{1}{4r^n} \iint_{E_r(t,x)} u(s,y) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds,$$

which implies

$$u(t_0, x_0) = \frac{1}{4r^n} \iint_{E_r(t,x)} u(s,y) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds,$$

and $u(s, y) = u(t_0, x_0)$ for all $y \in E_r(t_0, x_0)$. Consider $S = \{(s, y) \in (0, T] \times \Omega : u(s, y) = u(t_0, x_0)\}$, then by the above discussion, S is open in $(0, T] \times \Omega$. In addition, because u is continuous, then S is also closed in $(0, T] \times \Omega$. Since $(0, T] \times \Omega$ is connected, therefore $S = (0, T] \times \Omega$, which is a contradiction! Therefore $(t_0, x_0) \in [0, T] \times \overline{\Omega}/(0, T] \times \Omega$.

(3) Since u is a subsolution of the heat equation, by (2),

$$\max_{[0,T]\times\overline{\Omega}} u = \max\{\max_{0<x<1} \sin \pi x, \max_{t>0} 2te^{1-t}, \max_{t>0}(1 - \cos \pi t)\} = 2.$$